The far-side location of the cathode root could be due to the streamwise component of magnetic induction; this component is of opposite sense on opposite sides of the electrode plane, and the Lorentz force on the cathode root would always be toward the far side of the cathode. A similar effect might also be expected from the self-magnetic field of the curved arc root.

The evidence is strong that magnetic confinement for the stable arc is determined by dynamic processes in the positive column: 1) streamwise arc location along the rails changes with field-coil location as well as with freestream conditions; 2) arc location is not affected significantly by sharp ridges cut into the cathode material, or by $\frac{1}{16}$-in. flow baffles placed upstream of the cathode root; and 3) arc column location (and therefore stabilizing induction) is essentially the same for copper and carbon electrodes and for 0.6- and 1.1-in. interelectrode spacings.

There is an apparent disparity between these results and the results of rail-accelerator experiments in still air,[6, 7] since the rail-accelerator experiments indicate that the moving arc is dominated by cathode root phenomena rather than by processes in the column.[7] This disparity is probably due simply to the fact that with the present setup there is no requirement for motion of the cathode spot over the cathode surface as there is with the rail accelerator.

In addition to the stable mode of arc confinement described previously, a fluctuating mode was observed whenever conditions were mismatched such that a root station for the stable arc would have fallen at or beyond the upstream or downstream end of the rail (cylinder). This mode is manifest by wild spatial fluctuations of the column and by concomitant fluctuations in arc voltage. These fluctuations evidently result not from any intrinsic instability in the column itself, but from root constraints that interfere with the stable column configuration.

The observation that root constraints may cause column fluctuation suggests that the fluctuation observed[3–5] with the rail accelerator may have resulted from constraints imposed on column motion by the requirements for root motion, e.g., root motion in the stepping mode. If this were so, then one might expect that, with the removal of the requirement for root motion, a dramatic increase in column stability might occur, as is observed in the present experiment.

Conclusions

1) It is possible to magnetically confine within a supersonic airstream, a stable arc discharge sustained by an electric field essentially normal to the flow vector. Confinement results from the use of rail electrodes and a transverse, externally applied magnetic field with monotonic increase in flux density from electrode tip to electrode base.

2) When conditions are such that the arc is held between the rails at streamwise locations sufficiently remote from the rail ends, the positive column is characterized by high spatial stability.

(3 Confinement in the stable mode is determined by dynamic processes in the positive column and is independent of material or flow conditions at the surface of the cathode. In this mode, the arc column is confined between the rails in a region where the electric field is essentially two-dimensional. The streamwise location of the arc is determined by the streamwise location of the external field coils, by the field coil current, and by the freestream flow conditions.

4) The stably confined arc is characterized by a well-defined luminous column, with concentrated root marks on the electrodes, rather than by a discharge sheet extending in the streamwise direction.

5) In Mach 2.5 air flow with a freestream stagnation temperature of about 530°R, the average magnetic induction required for stable confinement of a 300-amp-d.c. arc varies with freestream stagnation pressure from about 1900 gauss at 10-in. Hg to about 3500 gauss at 25-in. Hg.

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Trajectory Extrapolation in a Central Force Field

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Introduction

THE path of an orbiting vehicle may be described analytically by its deviation from a nominal path. To establish a nominal path, we assume a central force field and extrapolate motion from the vehicle's current state. Our procedure yields a set of integration constants that aid extrapolation and lead to a simple description of path deviations. In this respect, we assume that the force field is indeed central; hence path deviations arise exclusively from small errors in the vehicle's current state. As in Ref. 1, a set of error coefficients is obtained by partial differentiation of the integrals of the equations of motion. If the force field is only approximately central, path deviations also arise from errors distributed along the path. The influence of distributed error can be approximated by integrating a system of linear differential equations, a technique not pursued here but described in Ref. 2.

State Extrapolation

The following variables describe the vehicle's current and terminal states:

r,R = radial distances from the attractive center
u,U = transverse velocities
v,V = radial velocities
T = time difference between the current and terminal states
θ = centric range angle between the current and terminal states

where r, u, and v refer to current state and R, U, and V to terminal state. It is clear that both T and θ decrease monotonically to zero during the transformation from the current to the terminal state. Thus, either T or θ can serve as an

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independent variable. For analytic convenience, θ is our choice.

If G is the gravitational constant, state transformation is described by

$$r' = -rv/u \qquad v' = -u + G/ru \qquad u' = v \qquad T' = r/u \tag{1}$$

where (′) denotes differentiation with respect to θ. To predict R, U, V, and T as functions of r, u, v, and θ we write

$$\begin{aligned} R' &= 0 & R(r,u,v;0) &= r \\ U' &= 0 & U(r,u,v;0) &= u \\ V' &= 0 & V(r,u,v;0) &= v \\ T' &= r/u & T(r,u,v;0) &= 0 \end{aligned} \tag{2}$$

The boundary conditions state that the current and terminal states are identical when $\theta = 0$.

Solving for T, we are led directly to the solutions for R, U, and V. To this end, we exp ressthe total derivative T' in terms of its partial derivatives to obtain

$$(rv/u)T_r - vT_u + (u - G/ru)T_v - T_\theta = -r/u \tag{3}$$

An equivalent system of ordinary differential equations is

$$(-u/rv)dr = (1/v)du = ru(G - ru^2)^{-1}dv = d\theta = (u/r)dT \tag{4}$$

Solutions of (4) yields four invariants which are fundamental to extrapolation. The first and second terms of (4) lead to the equation

$$ru = H \qquad H \text{ const} \tag{5}$$

which states that angular momentum is conserved in a central force field. The first and third terms together with (5) lead to

$$H^2/r^2 - 2G/r + v^2 = E \qquad E \text{ const} \tag{6}$$

This equation states that total energy is conserved. The first and fourth terms together with (5) and (6) yield

$$\begin{aligned} \theta + \sin^{-1}[(G - H^2/r)/Z] &= A & E &\neq 0 \\ \theta + 2\tan^{-1}[(2Gr - H^2)^{1/2}/H] &= A & E &= 0 \end{aligned} \tag{7}$$

where $Z = (G^2 + EH^2)^{1/2}$. The implication in (7) is that the angle between the terminal radius and a reference axis is the constant A. The reference axis is defined subsequently. Finally, the first and fifth terms and Eq. (6) lead to

$$\left.\begin{aligned} &T + \{rv + G(-E)^{-1/2}\sin^{-1}[(G + Er)/Z]\}/E = B & E < 0 \\ &T + (Gr + H^2)(2Gr - H^2)^{1/2}/3G^2 = B & E = 0 \\ &T + [rv - GE^{-1/2}\log(rv + rE^{1/2} + GE^{-1/2})]/E = B & E > 0 \end{aligned}\right\} \tag{8}$$

Each term in (8) is the time required to traverse an angle in (7). Thus T is associated with θ, the constant B with A, and the remaining association is clear.

We can express the general solution of (3) by an arbitrary functional relationship between the invariants H, E, A, and B. Similarly, we can show that R, U, and V are expressed by arbitrary functional relationships between the invariants H, E, and A. The structures of the functionals are determined by the boundary conditions in (2). In the usual fashion we arrive at

$$\begin{aligned} R &= H^2/(G - Z\sin A) & E &\neq 0 \\ R &= (H^2/2G)\sec^2(\tfrac{1}{2}A) & E &= 0 \end{aligned} \tag{9}$$

To express U, V, and T, we consider only elliptic paths. Since $-G/r < E < 0$ is applicable, we characterize an elliptic path by the values of the invariants

$$\begin{aligned} H &= ru \\ E &= u^2 + v^2 - 2G/r \\ A &= \theta + \sin^{-1}[(G - H^2/r)/Z] \\ B &= T + \{rv + G(-E)^{-1/2}\sin^{-1}[(G + Er)/Z]\}/E \end{aligned} \tag{10}$$

For all points on a trajectory the associated sets of values of r, u, v, and T yield the same values for H, E, A, and B. So must R, U, V, and $T = 0$, since this set also is associated with a point on the trajectory. Consequently, once R is known, the remaining unknowns are readily computed if we exploit this invariance. Thus,

$$\begin{aligned} R &= H^2/(G - Z\sin A) \\ U &= H/R \\ V &= \pm(E + 2G/R - U^2)^{1/2} \\ T &= B - \{rv + G(-E)^{-1/2}\sin^{-1}[(G + Er)/Z]\}/E \end{aligned} \tag{11}$$

Given r, u, v, and θ we compute H, E, and A using (10); then, R, U, and V using (11). Next we compute B by evaluating the last equation of (10) at the terminal state. Finally we compute T with the last equation of (11).

If we prefer to treat T rather than θ as the independent variable, we specify T and compute H, E, and B using (10). Expressing E and B in terms of R, V, and $T = 0$, we can eliminate V and then solve for R. This requires an iterative computational procedure. Now we can compute A, U, V, and θ.

To resolve sign and angular ambiguities, we assume that $0 \leqslant \theta \leqslant 2\pi$ and define

$$\alpha = \sin^{-1}[(G - H^2/r)/Z] \qquad \beta = \sin^{-1}[(G + rE)/Z] \tag{12}$$

We also choose α and β so that the following conditions are satisfied:

$$|\alpha|, |\beta| \leqslant \tfrac{1}{2}\pi \text{ if } v \geqslant 0 \qquad \tfrac{1}{2}\pi < |\alpha|, |\beta| < \pi \text{ if } v < 0 \tag{13}$$

We observe in (9) for $E \neq 0$ and $|A| < 2\pi$ that $R(A) = R(A + \pi)$ if and only if $A = 0$. This means that A is measured from an axis that is normal to the axis of symmetry of the ellipse. We choose A to satisfy

$$A = \theta + \alpha - n\pi \qquad \begin{cases} n = 0 \text{ if } 0 \leqslant \theta + \alpha \leqslant \pi \\ n = 2 \text{ if } \pi < \theta + \alpha \end{cases} \tag{14}$$

The sign of V is determined from

$$V \geqslant 0 \text{ if } |A| \leqslant \tfrac{1}{2}\pi \qquad V < 0 \text{ if } \tfrac{1}{2}\pi < |A| < \pi \tag{15}$$

Finally, the equation for B in (11) leads to

$$ET = RV - rv + G(-E)^{-1/2}[\beta(R) - \beta(r) + m\pi] \tag{16}$$

$$m = 0 \text{ if } \beta(R) - \beta(r) < 0 \qquad m = 2 \text{ if } \beta(R) - \beta(r) > 0$$

The orbital period is defined by $r = R$, $v = V$ and $m = 2$.

Error Extrapolation—Elliptic Paths

The errors dR, dU, dV, and dT associated with R, U, V, and T, respectively, are represented by the following matrix-vector equation:

$$\begin{bmatrix} dR \\ dU \\ dV \\ dT \end{bmatrix} = \begin{bmatrix} R_r & R_u & R_v & R_\theta \\ U_r & U_u & U_v & U_\theta \\ V_r & V_u & V_v & V_\theta \\ T_r & T_u & T_v & T_\theta \end{bmatrix} \begin{bmatrix} dr \\ du \\ dv \\ d\theta \end{bmatrix} \tag{17}$$

The error coefficients which comprise the matrix are partial derivatives of the expressions in (11).

The general coefficients listed here apply if $H \neq 0$, $v \neq 0$, and $V \neq 0$

$$\left.\begin{aligned}
R_r &= R^2[(v/H)\sin\theta - (1/r)\cos\theta + 2/R]/r\\
R_u &= R^2[(v/u)\sin\theta - 2\cos\theta + 2r/R]/H\\
R_v &= R^2 \sin\theta/H\\
R_\theta &= R^2[(1/r - G/H^2)\sin\theta + v\cos\theta/H]\\
U_r &= u[(1/r)\cos\theta - (v/H)\sin\theta - 1/R]\\
U_u &= 2\cos\theta - (v/u)\sin\theta - r/R\\
U_v &= -\sin\theta\\
U_\theta &= (G/H - u)\sin\theta - v\cos\theta\\
V_r &= G\sin\theta/rH\\
V_u &= (1 + G/uH)\sin\theta\\
V_v &= \cos\theta\\
V_\theta &= (u - G/H)\cos\theta - v\sin\theta\\
T_r &= T_R R_r + T_Z Z_r + (2G/r^2)T_E + (RV_r - v + G/rv)/E\\
T_u &= T_R R_u + T_Z Z_u + 2uT_E + RV_u/E\\
T_v &= T_R R_v + T_Z Z_v + 2vT_E + (RV_v - r)/E\\
T_\theta &= T_R R_\theta + RV_\theta/E
\end{aligned}\right\} \quad (18)$$

The time-to-go coefficients contain the following terms not already defined:

$$\begin{aligned}
T_R &= [V - G/RV]/E\\
T_E &= [RV - rv - 3ET + 2G(1/v - 1/V)]/2E^2\\
T_Z &= [(G + RE)/RV - (G + rE)/rv][G/ZE^2]\\
Z_r &= H^2(E + G/r)/rZ\\
Z_u &= Hr(E + u^2)/Z\\
Z_v &= H^2v/Z
\end{aligned} \quad (19)$$

In addition to errors confined to the orbital plane, there are the angular errors subtended at the attracting center by the current and terminal cross-plane displacements. We label these $d\gamma$ and $d\Gamma$, respectively. The associated rotations about the initial and terminal radius vectors, are $d\lambda$ and $d\Lambda$. These errors are related by

$$d\Gamma = d\gamma\cos\theta + d\lambda\sin\theta \qquad d\Lambda = -d\gamma\sin\theta + d\lambda\cos\theta \quad (20)$$

It follows that the respective magnitudes dS and dW of the total terminal position and velocity errors satisfy

$$\begin{aligned}
(dS)^2 &= (dR)^2 + (Rd\theta)^2 + (Rd\Gamma)^2\\
(dW)^2 &= (dU)^2 + (dV)^2 + (Ud\Lambda)^2
\end{aligned} \quad (21)$$

In (17), we equate $d\theta$ to zero if we are interested in the errors when $\theta = 0$. However, if the mission is timed by a chronometer, dT is zero. It follows that

$$T_r dr + T_u du + T_v dv + T_\theta d\theta = 0 \quad (22)$$

and $d\theta$ can be computed in terms of dr, du, and dv.

A convenient partial check on the partial derivatives is to show that they satisfy the total derivatives

$$R' = U' = V' = H' = E' = A' = B' = Z' = 0 \quad (23)$$

These identities express the fact that R, U, V, H, E, A, B, and Z are invariant in the presence of the evolutionary changes dr, du, and dv arising from a change $d\theta$.

Special Cases

To resolve the indeterminancies that occur when either v or V is zero, it is convenient to refer to the following alternate forms for the partial derivatives of T:

$$\begin{aligned}
ET_r &= VR_r + RV_r + QE_r + G(-E)^{-1/2}[X_r(1 - X^2)^{-1/2} - Y_r(1 - Y^2)^{-1/2}] - v\\
ET_u &= VR_u + RV_u + QE_u + G(-E)^{-1/2}[X_u(1 - X^2)^{-1/2} - Y_u(1 - Y^2)^{-1/2}]\\
ET_v &= VR_v + RV_v + QE_v + G(-E)^{-1/2}[X_v(1 - X^2)^{-1/2} - Y_v(1 - Y^2)^{-1/2}] - r\\
ET_\theta &= VR_\theta + RV_\theta + G(-E)^{-1/2}X_\theta(1 - X^2)^{-1/2}
\end{aligned} \quad (24)$$

where $Q = \frac{1}{2}[-3T + (RV - rv)/E]$ and

$$\begin{aligned}
X &= \sin\beta(R) = (G + ER)/Z\\
Y &= \sin\beta(r) = (G + Er)/Z
\end{aligned} \quad (25)$$

If the initial or terminal state is one of the apsides, then

$$\begin{aligned}
Z &= \pm(G - H^2/R) = \mp(G + ER) \qquad V = 0\\
Z &= \pm(G - H^2/r) = \mp(G + Er) \qquad v = 0
\end{aligned} \quad (26)$$

In these equations, the upper sign refers to apocenter and the lower sign to pericenter. Two equations useful for simplifying (24) are

$$\begin{aligned}
(1 - X^2)^{1/2} &= RV(-E)^{1/2}/Z\\
(1 - Y^2)^{1/2} &= rv(-E)^{1/2}/Z
\end{aligned} \quad (27)$$

With the aid of Eqs. (22, 25, and 26) we can show that all the partial derivatives of X are zero if $V = 0$, and all those of Y are zero if $v = 0$. Since either $X^2 = 1$ or $Y^2 = 1$, each equation in (24) contains indeterminate forms that must be evaluated. For this purpose we define

$$J_r^2 = \lim_{A\to 1/2\pi}[X_r^2(1 - X^2)^{-1}] \qquad K_r^2 = \lim_{\alpha\to 1/2\pi}[Y_r^2(1 - Y^2)^{-1}] \quad (28)$$

with similar definitions for the remaining indeterminate forms in (24). After two applications of L'Hospital's rule, we arrive at

$$\begin{aligned}
J_r^2 &= -\left(\frac{dX_r}{dA}\right)^2\left(\frac{d^2X}{dA^2}X\right)^{-1}\\
K_r^2 &= -\left(\frac{dY_r}{d\alpha}\right)^2\left(\frac{d^2Y}{d\alpha^2}Y\right)^{-1}
\end{aligned} \quad (29)$$

Differentiating X and Y in (25), we see that at the apsides

$$\left.\begin{aligned}
J_r &= (-E)^{1/2}RV_r/(G - HU)\\
J_u &= (-E)^{1/2}RV_u/(G - HU)\\
J_v &= (-E)^{1/2}RV_v/(G - HU)\\
J_\theta &= (-E)^{1/2}RV_\theta/(G - HU)\\
K_r &= 0 \qquad K_u = 0\\
K_v &= (-E)^{1/2}r/(G - Hu)
\end{aligned}\right\} \quad (30)$$

With these results, we can express the error coefficients for the special cases. We first consider the case in which the terminal state, but not the current state, is one of the apsides. Since $V = 0$, A and E in (10) yield

$$\cos\theta = (G - H^2/r)/(G - H^2/R) \qquad \sin\theta = vH/(G - H^2/R) \quad (31)$$

The coefficients for $H \neq 0$, $v \neq 0$, and $V = 0$ are

$$\left.\begin{aligned}
R_r &= \frac{(GR^2/r^2 - Hu)}{(G - HU)} & U_r &= \frac{u}{R} - \frac{u(G - rU^2)}{r(G - HU)}\\
R_u &= \frac{u(R^2 - r^2)}{(G - HU)} & U_u &= \frac{r}{R} - \frac{U(Ru - rU)}{(G - HU)}\\
R_v &= \frac{R^2v}{(G - HU)} & U_v &= \frac{-Hv}{(G - HU)}\\
R_\theta &= 0 & U_\theta &= 0\\
V_r &= \frac{(Gv/r)}{(G - HU)} & E^2T_r &= (RE - G)V_r - v\left(E + \frac{G}{r}\right) - \frac{3GET}{r^2}\\
V_u &= \frac{(Hv + Gv/u)}{(G - HU)} & E^2T_u &= (RE - G)V_u - v\left(H + \frac{G}{u}\right) - 3uET\\
V_v &= \frac{(G - Hu)}{(G - HU)} & E^2T_v &= (RE - G)V_v - r(E + v^2) - 3vET + G\\
V_\theta &= U - \frac{G}{H} & E^2T_\theta &= (RE - G)V_\theta + \frac{G^2}{H}
\end{aligned}\right\} \quad (32)$$

We now postulate that $H \neq 0$, $V \neq 0$ and $v = 0$. The partial derivatives of R, U, and V are easily determined by substituting $v = 0$ in (18). Those of T are

$$\left.\begin{aligned} ET_r &= (V - G/RV)R_r + RV_r + 2G(Q - G/VE)/r^2 + GXZ_r/RVE \\ ET_u &= (V - G/RV)R_u + RV_u + 2u(Q - G/VE) + GXZ_u/RVE \\ ET_v &= (V - G/RV)R_v + RV_v - r - Gr/(G - H^2/r) \\ ET_\theta &= (V - G/RV)R_\theta + RV_\theta \end{aligned}\right\} \quad (33)$$

If both the initial and terminal states are apsides, two cases arise. If $\theta = \pi$ and $V = v = 0$

$$\left.\begin{aligned} &R_r = R^2(4G/H^2 - 1/r)/r \qquad R_u = R^2(4G/uH^2) \\ &R_v = R_\theta = 0 \qquad U_r = u/R - (4G/H - u)/r \\ &U_u = r/R - 4G/uH \qquad U_v = U_\theta = 0 \\ &V_\theta = G/H - u \qquad V_v = -1 \qquad V_u = V_r = 0 \\ &T_r = 3GT/r^2E \qquad T_\theta = R/U \\ &T_u = -3uT/E \qquad T_v = (r^2 - R^2)/(G - Hu) \end{aligned}\right\} \quad (34)$$

On the other hand, if $\theta = 2\pi$ and $V = v = 0$,

$$\begin{aligned} &R_r = U_u = V_v = 1 \qquad V_\theta = u - G/H \\ &T_r = 3GT/R^2E \qquad T_u = -3uT/E \\ &T_v = 0 \qquad T_\theta = r/u \end{aligned} \quad (35)$$

and the remaining partial derivatives are zero.

Discussion

Our representation of error in (17) is referenced to a fixed θ. If instead, we treat T as the independent variable the error coefficients can be expressed in terms of those in (17) as follows

$$\begin{bmatrix} dR \\ dU \\ dV \\ d\theta \end{bmatrix} = (M - N) \begin{bmatrix} dr \\ du \\ dv \\ dT \end{bmatrix} \qquad M = \begin{bmatrix} R_r & R_u & R_v & 0 \\ U_r & U_u & U_v & 0 \\ V_r & V_u & V_v & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$N = \frac{1}{T_\theta} \begin{bmatrix} R_\theta T_r & R_\theta T_u & R_\theta T_v & -R_\theta \\ U_\theta T_r & U_\theta T_u & U_\theta T_v & -U_\theta \\ V_\theta T_r & V_\theta T_u & V_\theta T_v & -V_\theta \\ T_r & T_u & T_v & -1 \end{bmatrix} \quad (36)$$

We now can systematically generate errors associated with a fixed time-to-go T.

A related set of error coefficients is generated in Ref. 3. To compare the general coefficients on p. 1865 of this reference to ours in (18), it is necessary to recognize the following correspondence in notation: $r \rightarrow r_1$, $r \rightarrow R$, $u \rightarrow v_1 \cos\theta_1$, $U \rightarrow v \cos\theta$, $v \rightarrow v_1 \sin\theta_1$, $V \rightarrow v \sin\theta$, $\theta \rightarrow \eta_f$, $\alpha \rightarrow \eta_1 - \frac{1}{2}\pi$, $A \rightarrow \eta - \frac{1}{2}\pi$, $G \rightarrow K$, $Z \rightarrow K\epsilon$, $E \rightarrow K(A - 2)/r_1$.

It is evident upon comparison that the state extrapolation equations in (10) and (11) have led to a simpler set of partial derivatives. Although we must compute the invariants and the terminal variables to enjoy this simplicity, similar computations also are required to convert the normalized expressions in Ref. 3 to their useful dimensional forms. Thus the simplicity referred to is not because of notation alone.

The information contained in Eq. (A18) on p. 1865 of Ref. 3 corresponds to partial derivatives of θ in the matrix $(M-N)$ of (36). However, there is no expression in Ref. 3 equivalent to $\theta_T = -1/T_\theta$. As a consequence, this reference contains no explicit coefficients corresponding to the partial derivatives of T.

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Longitudinal Flow over a Circular Cylinder with Surface Mass Transfer

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Consideration is given to axisymmetric laminar boundary-layer flow longitudinal to a circular cylinder with continuously-distributed mass injection (blowing) or mass removal (suction) at the surface. Such a flow differs from that along a flat plate because of the transverse curvature of the cylindrical surface. The transverse curvature effect gives rise to nonsimilar boundary-layer solutions.

The present analysis is carried out for constant property, incompressible flow. In the case of mass addition, the properties of the injected gas are identical to those of the mainstream gas. The cylinder has a radius r_0. Radial distances are measured by r, whereas x measures axial distances downstream from the effective starting point of the boundary layer. The freestream velocity is U_∞. Solutions will be sought for two distributions of the surface mass transfer velocity: 1) $v_w \sim x^{-1/2}$, and 2) $v_w = \text{const}$. The first of these is widely employed in mass transfer analyses, since it leads to similarity solutions for the flat-plate boundary layer. The second is of interest inasmuch as it is more easily achieved in practice. Other investigations concerned with the simultaneous effects of mass transfer and transverse curvature are reported in Refs. 1 and 2.

Analysis

1. $v_w \sim x^{-1/2}$

The analytical formulation for this case closely parallels that for the impermeable wall.[3] Upon satisfying mass conservation and transforming the momentum equation, there is obtained a partial differential equation for the dimensionless stream function f that cannot be made to yield similarity solutions. Recourse is then had to a perturbation series

$$f(\xi, \eta) = f_0(\eta) + \xi f_1(\eta) + \xi^2 f_2(\eta) + \xi^3 f_3(\eta) + \ldots \quad (1)$$

wherein

$$\begin{aligned} \xi &= (4/r_0)(\nu x/U_\infty)^{1/2} \\ \eta &= (U_\infty/\nu x)^{1/2}[(r^2 - r_0^2)/4r_0] \end{aligned} \quad (2)$$

In Ref. 3, the foregoing series was truncated at f_2, whereas in the present study terms up to and including f_3 have been retained. The ordinary differential equations governing f_0, f_1, and f_2 are stated in the reference and need not be repeated here. The f_3 equation is derived in a straightforward manner.

The essential modification in the analysis that stems from the $x^{-1/2}$ distribution of the surface mass transfer is a change in the boundary conditions. In particular, whereas formerly $f_0(0) = 0$, now $f_0(0) = F_w$, wherein

$$F_w = -2(v_w/U_\infty)(U_\infty x/\nu)^{1/2} \quad (3)$$

For the situation in which $v_w \sim x^{-1/2}$, it is evident that F_w is a constant parameter. Aside from the change in $f_0(0)$, all other boundary conditions remain unchanged. The nature of the governing equations is such that the solution for f_1 depends upon f_0, the solution for f_2 depends upon f_0 and f_1, and so forth. Thus, the fact that the f_0 function corresponding to surface mass transfer is different from the f_0 function

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